# Exercise 7.2.10

A certain differential equation has the form

$$f(x)\,dx + g(x)h(y)\,dy = 0,$$

with none of the functions $f(x)$, $g(x)$, $h(y)$ identically zero. Show that a necessary and sufficient condition for this equation to be exact is that $g(x) = \text{constant}$.

---

**Solution**

Suppose first that the ODE is exact. Then there exists a potential function $\varphi = \varphi(x, y)$ that satisfies

$$\frac{\partial \varphi}{\partial x} = f(x) \quad (1)$$

$$\frac{\partial \varphi}{\partial y} = g(x)h(y). \quad (2)$$

Substitute these formulas into the ODE.

$$\frac{\partial \varphi}{\partial x}\,dx + \frac{\partial \varphi}{\partial y}\,dy = 0$$

On the left is how the differential of $\varphi$ is defined.

$$d\varphi = 0$$

Integrate both sides.

$$\varphi(x, y) = C$$

The solution to the ODE is found then by solving equations (1) and (2) for $\varphi$. Start by integrating both sides of equation (1) partially with respect to $x$.

$$\varphi(x, y) = \int^x f(r)\,dr + F(y)$$

$F(y)$ here is an arbitrary function of $y$, and the lower limit of integration is arbitrary as well. Differentiate both sides with respect to $y$.

$$\begin{aligned}
\frac{\partial \varphi}{\partial y} &= \frac{\partial}{\partial y}\left[\int^x f(r)\,dr + F(y)\right] \\
&= \frac{\partial}{\partial y}\int^x f(r)\,dr + F'(y) \\
&= F'(y)
\end{aligned}$$

Comparing this formula for $\partial\varphi/\partial y$ to equation (2), we see that

$$F'(y) = g(x)h(y).$$

In order to integrate both sides with respect to $y$ and solve for $F(y)$, the function $g(x)$ must be a constant.

In order for the ODE,

$$f(x)\,dx + g(x)h(y)\,dy = 0,$$

to be exact, it must satisfy

$$\frac{\partial}{\partial y}[f(x)] = \frac{\partial}{\partial x}[g(x)h(y)]$$
$$0 = h(y)\frac{dg}{dx}.$$

Since $h(y) \neq 0$, divide both sides by $h(y)$.

$$\frac{dg}{dx} = 0$$

Integrate both sides with respect to $x$.

$$g(x) = \text{constant}$$

Therefore, a necessary and sufficient condition for this equation to be exact is that $g(x) = \text{constant}$.